

COMMENSURABILITY AND REPRESENTATION EQUIVALENT ARITHMETIC LATTICES

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ABSTRACT. In ([PR]), Gopal Prasad and A. S. Rapinchuk defined a notion of weakly commensurable lattices in a semisimple group, and gave a classification of weakly commensurable Zariski dense subgroups. A motivation was to classify pairs of locally symmetric spaces isospectral with respect to the Laplacian on functions. For this, in higher ranks, they assume the validity of Schanuel's conjecture.

In this note, we observe that if we use the stronger notion of representation equivalence of lattices, then Schanuel's conjecture can be avoided. Further, the results are also applicable in a S -arithmetic setting.

We also introduce a new relation on the class of arithmetic lattices, stronger than weak commensurability, which we call as characteristic equivalence, and show that it simplifies some of the arguments used in [PR] to deduce commensurability type results from weak commensurability.

1. INTRODUCTION

Let M be a compact, connected Riemannian manifold. The spectrum of the Laplace-Beltrami operator acting on the space of smooth functions on M , the collection of its eigenvalues counted with (finite) multiplicity, is a discrete weighted subset of the non-negative reals. Define two compact connected Riemannian manifolds M_1 and M_2 to be *isospectral on functions* or just *isospectral*, if the spectra of the Laplace-Beltrami operator acting on the space of smooth functions on M_1 and M_2 coincide.

The inverse spectral problem is to recover the properties of the Riemannian manifold M from a knowledge of the spectrum. It is known, for example, that the spectra on functions determines the dimension, volume and the scalar curvature of M .

Milnor constructed the first examples in the context of flat tori of non-isometric compact Riemannian manifolds which are isospectral on functions. When the spaces are compact hyperbolic surfaces, such examples were initially constructed by Vigneras [V]. In analogy with a construction in arithmetic, Sunada gave a general method for constructing pairs of isospectral spaces [S].

In many of these constructions, the manifolds are quotients by finite groups of a fixed Riemannian manifold. The question arises whether isospectral manifolds are indeed commensurable, i.e., have a common finite cover. In the context of Riemannian locally symmetric spaces this question has been studied by various

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authors ([R, CHLR, PR, LSV]) assuming that the spaces are isospectral for the Laplace-Beltrami operator acting on functions. Gopal Prasad and A. S. Rapinchuk address this question in full generality, and get commensurability type results for isospectral, compact locally symmetric spaces. For this when the locally symmetric spaces are of rank at least two, they have to assume the validity of Schanuel's conjecture on transcendental numbers.

In this note, we consider this question assuming a stronger hypothesis that the lattices defining the locally symmetric spaces are representation equivalent rather than isospectral on functions. This allows us to derive the results of [PR], without invoking Schanuel's conjecture, and also extend the application to representation equivalent S -arithmetic lattices. In the process, we introduce a new relation of characteristic equivalence of lattices, stronger than weak commensurability. This helps in simplifying some of the arguments used in [PR].

2. REPRESENTATION EQUIVALENCE OF LATTICES

The Fourier analysis for the circle group S^1 can be studied in two ways: either, as expanding a function in terms of the eigenfunctions of the Laplace operator, or via characters of the topological group S^1 . In the context of Riemannian locally symmetric spaces the spectrum can also be studied in terms of representation theory of the isometry group of the universal cover.

Let G be a locally compact, unimodular topological group and Γ be a uniform lattice in G . Let R_Γ denote the right regular representation of G on the space $L^2(\Gamma \backslash G)$ of square integrable functions with respect to the projection of the Haar measure on the space $\Gamma \backslash G$:

$$(R_\Gamma(g)f)(x) = f(xg), \quad f \in L^2(\Gamma \backslash G), \quad g, x \in G.$$

As a G -space, $L^2(\Gamma \backslash G)$ breaks up as a direct sum of irreducible unitary representations of G ,

$$L^2(\Gamma \backslash G) \simeq \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) \pi,$$

where \hat{G} is the unitary dual of G parametrizing isomorphism classes of irreducible, unitary representations of G , and $m(\pi, \Gamma)$ is the (finite) multiplicity with which an element $\pi \in \hat{G}$ occurs in $L^2(\Gamma \backslash G)$. Define the *representation spectrum* of a uniform lattice $\Gamma \subset G$ to be the map $\pi \mapsto m(\pi, \Gamma)$ giving the multiplicity $m(\pi, \Gamma)$ with which an irreducible unitary representation π of G occurs in $L^2(\Gamma \backslash G)$.

Definition 2.1. Let G be a locally compact topological group and Γ_1 and Γ_2 be two uniform lattices in G . The lattices Γ_1 and Γ_2 are said to be *representation equivalent* in G if

$$L^2(\Gamma_1 \backslash G) \cong L^2(\Gamma_2 \backslash G)$$

as G -spaces.

The relevance of this notion to spectrum is provided by the following generalization of Sunada's criterion for isospectrality [DG]:

Theorem 2.2. *Let G be a locally compact topological group G which acts on a Riemannian manifold M . Let Γ_1, Γ_2 be representation equivalent uniform lattices in G . Suppose G acts on a Riemannian manifold M , such that Γ_1, Γ_2 act properly discontinuously and freely on M with compact quotients. Then the Riemannian manifolds $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ with respect to the induced metric from M are strongly isospectral; in particular, they are isospectral on p -forms for all p .*

The concept of strong isospectrality is defined in [DG] as having the same spectrum for any natural (in the sense of Epstein and Stredder) elliptic differential operator with positive definite symbol. A plausible alternate definition is as follows: suppose two compact oriented Riemannian manifolds M, N are isospectral on functions. Then it is known that their dimensions are equal, say of dimension d . The Riemannian metric gives a reduction of structure group of the tangent bundle to the orthogonal group $SO(d)$. Given a representation τ of $SO(d)$, this defines two metrized vector bundles on M and N respectively. A Laplace type operator (elliptic, self-adjoint, non-negative) can be defined on the space of smooth sections of these bundles. For strongly isospectral, we require that for any τ as above, these Laplace operators have the same spectrum. For example, one can consider the spectrum of the Hodge-deRham Laplacians acting on the space of smooth p -forms of a oriented compact Riemannian manifold.

Suppose $M = G/K$ is a noncompact Riemannian symmetric space, where G is a noncompact semisimple Lie group and K is a maximal compact subgroup of G . Let Γ be a uniform torsion-free lattice in G . To an irreducible representation τ of K there is associated an automorphic vector bundle E_τ on the quotient space $\Gamma \backslash G/K$. The above theorem implies that if the lattices are representation equivalent, then the spectra of the Laplace operators on the smooth sections of E_τ are equal.

Remark 2.3. In [P], Pesce has proved that the converse of the generalized Sunada Criterion holds in the case of $G = \text{Isom}(\mathbb{H}^n)$, where \mathbb{H}^n is the hyperbolic n -space with constant sectional curvature -1 . However, in the general context of locally symmetric spaces, the converse to the generalized Sunada criterion is not known, i.e, whether isospectrality for all automorphic vector bundles as above yields representation equivalence.

For compact hyperbolic surfaces X , it is known that the spectrum on functions determines the representation equivalence of the lattice $\pi_1(X, x_0) \subset PSL(2, \mathbb{R})$. This prompts the following question:

Question 2.4. Will it be true that for compact quotients of non-compact Riemannian symmetric spaces, the spherical spectrum (the restriction of the representation spectrum to the class of spherical representations of G) determines

the representation class of the lattice in the group of isometries ([BR])? More generally, will this be true if we just look at the spectrum of the Laplacian on functions?

When the lattices are in different groups, we can compare them only upto a topological isomorphism:

Definition 2.5. Let G_1, G_2 be locally compact topological groups and Γ_1 (resp. Γ_2) be uniform lattices in G_1 (resp. G_2). The lattices Γ_1 and Γ_2 are said to be *topologically representation equivalent* if there exists an isomorphism $\phi : G_1 \rightarrow G_2$ of topological groups such that

$$L^2(\phi(\Gamma_1) \backslash G_2) \cong L^2(\Gamma_2 \backslash G_2)$$

as G_2 -spaces.

2.1. Arithmetic lattices. We will have the following notations and assumptions for the rest of this paper:

H1: \mathcal{G} is a connected absolutely almost simple algebraic group defined over a number field K .

H2: S is a finite set of places of K containing the archimedean places at which \mathcal{G} is isotropic. Let S^i denote the subset of places of S at which \mathcal{G} is isotropic.

H3: There is at least one place $v \in S$ at which \mathcal{G} is isotropic.

A subgroup Γ of $\mathcal{G}(K)$ is said to be (\mathcal{G}, K, S) -arithmetic (or just arithmetic) subgroup, if it is commensurable with $\mathcal{G}(\mathcal{O}_K(S)) = \mathcal{G}(K) \cap GL_n(\mathcal{O}_K(S))$, where $\mathcal{O}_K(S)$ is the set of S -integers in K and we consider \mathcal{G} as embedded in GL_n over K for some n .

Denote by \mathcal{G}_S the locally compact group,

$$\mathcal{G}_S = \prod_{v \in S} \mathcal{G}(K_v),$$

where given a place v of K , K_v denotes the completion of K at v .

There is an embedding of the arithmetic subgroup $\Gamma \subset \mathcal{G}_S$, which is well defined upto complex conjugation at the complex places of K . By results of Borel, Harishchandra, Godement and Tamagawa, this defines an arithmetic lattice $\Gamma \subset \mathcal{G}_S$, which is Zariski dense in \mathcal{G} .

Suppose $\mathcal{G}_1, \mathcal{G}_2$ are algebraic groups as above, and assume further that are anisotropic. Then the lattices Γ_i are cocompact in \mathcal{G}_{i, S_i} for $i = 1, 2$. We define two S -arithmetic subgroups $\Gamma_1 \subset \mathcal{G}_1(K_1)$, $\Gamma_2 \subset \mathcal{G}_2(K_2)$ to be topologically representation equivalent if the lattices $\Gamma_1 \subset \mathcal{G}_{1, S_1}$, $\Gamma_1 \subset \mathcal{G}_{2, S_2}$ are topologically representation equivalent.

Remark 2.6. By theorems of Freudenthal and Borel-Tits [BT], it is known that any abstract homomorphism of adjoint Lie groups as above is automatically continuous. Hence in the definition of representation equivalence we could have

just required that there is an abstract isomorphism between the ambient groups, requiring that the image of the lattice Γ_1 is again a lattice (so that representation equivalence makes sense).

Denote by $\mathcal{G} \rightarrow \overline{\mathcal{G}}$ the isogeny to the adjoint group corresponding to \mathcal{G} . For a subgroup Γ of $\mathcal{G}(K)$, $\overline{\Gamma}$ will denote the image in $\overline{\mathcal{G}}(K)$.

Define two arithmetic subgroups $\Gamma_1 \subset \mathcal{G}_1(K_1)$, $\Gamma_2 \subset \mathcal{G}_2(K_2)$ to be *commensurable*, if there are isomorphisms $\sigma : \text{Spec} K_2 \rightarrow \text{Spec} K_1$ and $\phi : {}^\sigma \overline{\mathcal{G}}_1 \rightarrow \overline{\mathcal{G}}_2$, where the superscript σ denotes twisting the group scheme \mathcal{G}_1 by σ . In particular, the image $\phi(\overline{\Gamma}_1)$ considered as a subgroup of $\overline{\mathcal{G}}_2(K_2)$ and $\overline{\Gamma}_2$ will be commensurable subgroups.

2.2. Main Theorem. Working with representation equivalence of arithmetic lattices rather than isospectrality on functions of the corresponding locally symmetric space, allows us to establish the following results of Gopal Prasad and Rapinchuk unconditionally, without assuming the validity of Schanuel's conjecture on transcendental numbers:

Theorem 2.7. *Let \mathcal{G}_1 (resp. \mathcal{G}_2) be anisotropic algebraic groups defined over a number field K_1 (resp. K_2). Let S_1 (resp. S_2) be a finite set of places of K_1 (resp. K_2). Assume that for $i = 1, 2$, $(K_i, \mathcal{G}_i, S_i)$ satisfy hypothesis **H1-H3**.*

Let $\Gamma_1 \subset \mathcal{G}_1(K_1)$ (resp. $\Gamma_2 \subset \mathcal{G}_2(K_2)$) be S_1 (resp. S_2)-arithmetic subgroup of \mathcal{G}_1 (resp. \mathcal{G}_2).

Suppose that the lattices $\Gamma_1 \subset \mathcal{G}_{1,S_1}$, $\Gamma_2 \subset \mathcal{G}_{2,S_2}$ are topologically representation equivalent.

Then the following holds:

- (1) *The groups \mathcal{G}_1 and \mathcal{G}_2 are of the same geometric type, i.e., $\overline{\mathcal{G}}_1 \times \overline{K} \simeq \overline{\mathcal{G}}_2 \times \overline{K}$.*
- (2) *The fields K_1 and K_2 are Galois conjugate.*
- (3) *There exists an isomorphism $\sigma : K_1 \rightarrow K_2$ such that the set of isotropic places $S_1^i = \sigma^*(S_2^i)$.*
- (4) *If \mathcal{G}_1 is not of type A_n , D_4 , D_{2n+1} , E_6 ($n > 1$), then the lattices Γ_1 and Γ_2 are commensurable, i.e., $\overline{\mathcal{G}}_1 \simeq \overline{\mathcal{G}}_2$ over K .*
- (5) *In any topologically representation equivalence class of arithmetic lattices, there are only finitely many commensurability classes of arithmetic lattices.*

Part (1) of the above theorem, follows immediately from the definition of topologically representation equivalent lattices. The existence of an isomorphism between \mathcal{G}_{1,S_1} and \mathcal{G}_{2,S_2} gives an isomorphism at the level of Lie algebras. By assumption, at any place $v_1 \in S_1$ (resp. $v_2 \in S_2$), the Lie algebra of $\mathcal{G}_1(K_{1,v_1})$ (resp. $\mathcal{G}_2(K_{2,v_2})$) is simple. Hence (1) follows.

Remark 2.8. Since the lattices Γ_i are uniform for $i = 1, 2$, any element belonging to Γ_i is semisimple.

Remark 2.9. The first instance of this theorem was established by A. Reid [R], who showed that the spectrum on functions of an arithmetic compact hyperbolic surface associated to a quaternion division algebra defined over a totally real number field determines the underlying number field and the division algebra.

For a compact Riemannian manifold M , denote by $L(M)$ the subset of \mathbb{R} consisting of lengths of closed geodesics in M . Two Riemannian manifolds M_1 and M_2 are said to be length commensurable (resp. length isospectral) if $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$ (resp. $L(M_1) = L(M_2)$). The starting point of the proof of Reid's theorem is to use the Selberg trace formula to conclude that two compact hyperbolic surfaces are isospectral if and only if their length spectrums coincide.

Reid also proved that the complex length spectrum (length together with the holonomy of the closed geodesic) of an arithmetic hyperbolic three manifold determines the commensurability class of the manifold. It can be seen from the trace formula that the complex length spectrum determines the representation equivalence class of the lattice. Working with only the length spectrum, Chinburg, Hamilton, Long and Reid showed in [CHLR] that length commensurable hyperbolic three manifolds are commensurable.

These results were vastly generalized by Gopal Prasad and A. Rapinchuk ([PR]). First, using results of Duistermaat, Guillemin, Kolk and Varadarajan ([DG, DKV]), Prasad-Rapinchuk-Urbe-Zelditch show that if two compact, Riemannian locally symmetric spaces of nonpositive sectional curvature are isospectral for the Laplace-Beltrami operator on functions then they are length commensurable (see Theorem 10.1 in [PR]).

Prasad and Rapinchuk define a notion of weak commensurability of lattices (see Section 3.4). We recall Schanuel's conjecture on transcendental numbers:

Conjecture 2.10 (Schanuel). If z_1, \dots, z_n are \mathbb{Q} -linearly independent complex numbers, then the transcendence degree over \mathbb{Q} of the field generated by

$$z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}$$

is at least n .

Prasad and Rapinchuk assuming the validity of Schanuel's conjecture when the locally symmetric spaces are of rank greater than one, show that length commensurable arithmetic lattices are weakly commensurable,

Using methods from arithmetic theory of algebraic groups, they then conclude results on commensurability, in particular the conclusions of Theorem 2.7, from the notion of weak commensurability of lattices.

The use of representation equivalence instead of isospectrality on functions allows us to bypass the use of Schanuel's conjecture in the higher ranks. The proof of Theorem 2.7 is an application of the Selberg trace formula and the ideas and methods given in [PR].

Remark 2.11. An initial motivation for this paper was to extend the results of A. Reid [R] to the context of S -arithmetic groups. An advantage of working

with the representation theoretic spectrum, is that the notion applies even when there is no Riemannian geometric interpretation. This allows us to consider S -arithmetic lattices.

Remark 2.12. Examples of representation equivalent lattices which are not commensurable have been given by Lubotzky, Samuels and Vishne [LSV]. It would be interesting to know whether such examples can be constructed in the exceptional cases given in [PR, Section 9], where commensurability fails.

3. ELEMENT-WISE CONJUGATE LATTICES

Definition 3.1. Let G_1, G_2 be locally compact topological groups and Γ_1 (resp. Γ_2) be lattices in G_1 (resp. G_2). The lattices Γ_1 and Γ_2 are said to be topologically elementwise conjugate if there exists an isomorphism $\phi : G_1 \rightarrow G_2$ of topological groups such that for any element $\gamma_1 \in \Gamma_1$ (resp. $\gamma_2 \in \Gamma_2$) there exists an element $\gamma_2 \in \Gamma_2$ (resp. $\gamma_1 \in \Gamma_1$) such that $\phi(\gamma_1)$ and γ_2 are conjugate in G_2 .

An application of the Selberg trace formula for compact quotients yields the following theorem:

Theorem 3.2. *Let G_1, G_2 be locally compact topological groups of algebraic type, and Γ_1 (resp. Γ_2) be uniform lattices in G_1 (resp. G_2). Suppose the lattices Γ_1 and Γ_2 are topologically representation equivalent. Then they are topologically elementwise conjugate.*

3.1. Selberg trace formula. We recall the Selberg trace formula for uniform lattices [W]. Let f be a continuous, compactly supported function on G . The convolution operator $R_\Gamma(f)$ on $L^2(\Gamma \backslash G)$ is defined by,

$$R_\Gamma(f)(\phi)(x) = \int_G f(y) R_\Gamma(y)(\phi)(x) d\mu(y),$$

where μ is an invariant Haar measure on G . It is known that $R_\Gamma(f)$ is of trace class.

Let $[\gamma]_G$ (resp. $[\gamma]_\Gamma$) be the conjugacy class of γ in G (resp. in Γ). Let $[\Gamma]$ (resp. $[\Gamma]_G$) be the set of conjugacy classes in Γ (resp. the G -conjugacy classes of elements in Γ). For $\gamma \in \Gamma$, let G_γ be the centralizer of γ in G . Put $\Gamma_\gamma = \Gamma \cap G_\gamma$. It can be seen that Γ_γ is a lattice in G_γ and the quotient $\Gamma_\gamma \backslash G_\gamma$ is compact. Since G_γ is unimodular, there exists a G -invariant measure on $G_\gamma \backslash G$, denoted by $d_\gamma x$. After normalizing the measures on G_γ and $G_\gamma \backslash G$ appropriately and rearranging the terms on the right hand side of above equation, we get :

$$(1) \quad \text{tr}(R_\Gamma(f)) = \sum_{[\gamma] \in [\Gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) d_\gamma x$$

$$= \sum_{[\gamma] \in [\Gamma]_G} a(\gamma, \Gamma) O_\gamma(f)$$

where $O_\gamma(f)$ is the orbital integral of f at γ defined by,

$$O_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) \, d_\gamma x.$$

Here

$$a(\gamma, \Gamma) = \sum_{[\gamma']_\Gamma \subseteq [\gamma]_G} \text{vol}(\Gamma_{\gamma'} \backslash G_{\gamma'}).$$

If γ is not conjugate to an element in Γ , we define $a(\gamma, \Gamma) = 0$.

Let π be an irreducible unitary representation of G . Denote by $\chi_\pi(f)$ the distributional character of π given by,

$$\chi_\pi(f) = \text{Trace}(\pi(f)).$$

The trace of $R_\Gamma(f)$ on the spectral side can be written as an absolutely convergent series as,

$$(2) \quad \text{tr}(R_\Gamma(f)) = \sum_{\pi \in \widehat{G}} m(\pi, \Gamma) \chi_\pi(f)$$

Hence from (1) and (2), we obtain the Selberg trace formula:

$$(3) \quad \sum_{\pi \in \widehat{G}} m(\pi, \Gamma) \chi_\pi(f) = \sum_{[\gamma] \in [\Gamma]_G} a(\gamma, \Gamma) O_\gamma(f).$$

3.2. Proof of Theorem 3.2. We prove a few lemmas before giving the proof of Theorem 3.2.

Lemma 3.3. *Let G be a locally compact topological group and Γ be a uniform lattice in G . Let U be a relatively compact subset of G . Then the set*

$$A_U = \{ [\gamma]_G : \gamma \in \Gamma \text{ and } [\gamma]_G \cap U \neq \emptyset \}$$

is finite.

Proof. Since the quotient $\Gamma \backslash G$ is compact, there exists a relatively compact subset D of G such that $G = \Gamma D$. Let $x \in G$ be such that $x^{-1}\gamma x \in U$ for some $\gamma \in \Gamma$. Write $x = \gamma' \cdot \delta$ where $\gamma' \in \Gamma$ and $\delta \in D$. Hence $\gamma'^{-1}\gamma\gamma' \in DUD^{-1}$ which is relatively compact in G . Hence $\gamma'^{-1}\gamma\gamma' \in DUD^{-1} \cap \Gamma$ which is a finite set. \square

Lemma 3.4. *Let Γ_1 and Γ_2 be uniform lattices in G . Let $\gamma_1 \in \Gamma_1$. Then there exists a relatively compact open set U containing γ_1 such that*

$$U \cap [\gamma]_G = \emptyset$$

whenever $\gamma \in \Gamma_1 \cup \Gamma_2$ and $[\gamma_1]_G \neq [\gamma]_G$.

Proof. Easily follows from Lemma 3.3. \square

Proof of Theorem 3.2. By comparing the Selberg trace formula ((3)) for the lattices $\Gamma'_1 = \phi(\Gamma_1)$ and Γ_2 in $G = G_2$, we get for any compactly supported continuous function f on G ,

$$\sum_{\pi \in \widehat{G}} [m(\pi, \Gamma'_1) - m(\pi, \Gamma_2)] \chi_\pi(f) = \sum_{[\gamma] \in [\Gamma'_1]_G \cup [\Gamma_2]_G} [a(\gamma, \Gamma'_1) - a(\gamma, \Gamma_2)] O_\gamma(f).$$

Since the lattices Γ'_1, Γ_2 are representation equivalent in G , the left side is identically zero in the above equation.

Suppose $\gamma_1 \in \Gamma'_1$ is not conjugate to any element of Γ_2 in G . Choose U as in Lemma 3.4, and a positive function f supported on U . For such f , we have that the orbital integral $O_\gamma(f)$ vanishes whenever $[\gamma]_G \neq [\gamma_1]_G$. Further $O_{\gamma_1}(f)$ is non-zero.

It follows that all terms on the right hand side above vanish except that corresponding to $[\gamma_1]_G$. Consequently, $a(\gamma_1, \Gamma'_1) O_{\gamma_1}(f) = 0$. Since both these quantities are non-zero by definition, we arrive at a contradiction. Hence the lattices $\Gamma'_1 = \phi(\Gamma_1)$ and Γ_2 are elementwise conjugate in $G = G_2$. \square

3.3. Characteristic equivalence. Let \mathcal{G} be an algebraic group defined over a number field K . Consider the adjoint action Ad of \mathcal{G} on its Lie algebra $L(\mathcal{G})$. Given a semisimple element $\gamma \in \mathcal{G}(L)$ for an extension field L of K , and any field M containing L , denote by $P(Ad_{\mathcal{G}}(\gamma), x)$ the characteristic polynomial of $Ad(\gamma)$ acting on $L(\mathcal{G}) \otimes_K M$. The characteristic polynomial is independent of the extension field M , and has coefficients in L . The characteristic polynomial is also independent upto isomorphisms:

Lemma 3.5. *Let $\mathcal{G}_1, \mathcal{G}_2$ be simple algebraic groups defined over an algebraically closed field F , and let $\theta : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be an isomorphism defined over F . Suppose t is a semisimple element in $\mathcal{G}_1(F)$. Then*

$$P(Ad_{\mathcal{G}_1}(t, x)) = P(Ad_{\mathcal{G}_2}(\theta(t), x)).$$

Proof. Let T_1 be a maximal torus in \mathcal{G}_1 containing t . The eigenvalues of $Ad_{\mathcal{G}_1}(t)$ are 0 with multiplicity equal to the rank of \mathcal{G}_1 and $\alpha(t)$ where α runs over the roots of $L(\mathcal{G}_1)$ with respect to T_1 . If X_α is a root vector corresponding to the root α , then

$$Ad(\theta(t))(d\theta(X_\alpha)) = d\theta(Ad(t)X_\alpha) = \alpha(t)X_\alpha.$$

Hence the eigenvalues of $\theta(t)$ are the same as t , and this proves the lemma. \square

The topological elementwise conjugacy of the lattices Γ_1 and Γ_2 given by Theorem 3.2 yields the following key proposition stating an equality of characteristic polynomials with respect to the adjoint representation:

Proposition 3.6. *With assumptions as in Theorem 2.7, there exists a locally compact field F and embeddings $\iota_1 : K_1 \rightarrow F$, $\iota_2 : K_2 \rightarrow F$, and a topological automorphism σ of F such that given any element $\gamma_1 \in \Gamma_1$ (resp. $\gamma_2 \in \Gamma_2$) there exists an element $\gamma_2 \in \Gamma_2$ (resp. $\gamma_1 \in \Gamma_1$) such that the characteristic polynomials coincide,*

$$\sigma(P(\text{Ad}_{\mathcal{G}_1}(\gamma_1), x)) = P(\text{Ad}_{\mathcal{G}_2}(\gamma_2), x).$$

Proof. By theorems of E. Cartan for archimedean places and Bruhat-Tits for non-archimedean places, it is known that $\overline{\mathcal{G}}(K_v)$ is a simple group for any place v of K . By hypothesis of topological representation equivalence, we are given a topological isomorphism $\phi : \mathcal{G}_{1,S_1} \rightarrow \mathcal{G}_{2,S_2}$. It follows that for any place $v_1 \in S_1$ there exists a place $v_2 \in S_2$ such that ϕ induces an isomorphism (denoted again by ϕ)

$$\overline{\mathcal{G}}_1(K_{1,v_1}) \rightarrow \overline{\mathcal{G}}_2(K_{2,v_2}).$$

Let $\overline{\Gamma}_1$ (resp. $\overline{\Gamma}_2$) be the image of the projection to $\overline{\mathcal{G}}_1(K_{1,v_1})$ (resp. $\overline{\mathcal{G}}_2(K_{2,v_2})$) of the lattices Γ_1 (resp. Γ_2). By Theorem 3.2, given any element $\gamma_1 \in \Gamma_1$ (resp. $\gamma_2 \in \Gamma_2$) there exists an element $\gamma_2 \in \Gamma_2$ (resp. $\gamma_1 \in \Gamma_1$) such that the element $\phi(\overline{\gamma}_1)$ (resp. $\overline{\gamma}_2$) is conjugate in $\overline{\mathcal{G}}_2(K_{2,v_2})$ to $\overline{\gamma}_2$ (resp. $\phi(\overline{\gamma}_1)$).

By Borel-Tits, there is a continuous isomorphism $\sigma : K_{1,v_1} \rightarrow K_{2,v_2}$ such that the isomorphism ϕ is induced by an algebraic isomorphism between the base changed group schemes,

$${}^\sigma(\overline{\mathcal{G}}_1 \times K_{1,v_1}) \rightarrow \overline{\mathcal{G}}_2 \times K_{2,v_2},$$

where the superscript σ denotes twisting the group scheme \mathcal{G}_1 by σ .

Let $F = K_{2,v_2}$ and $\iota_2 : K_2 \rightarrow F$ be the natural embedding. The restriction of σ to K_1 gives an embedding ι_1 of K_1 into F . By Lemma 3.5, we have

$$P(\text{Ad}_{\mathcal{G}_2}(\gamma_2), x) = P(\text{Ad}_{\mathcal{G}_2}(\phi(\gamma_1)), x) = \sigma(P(\text{Ad}_{\mathcal{G}_1}(\gamma_1), x)).$$

This proves the proposition. \square

We now show Part (2) of Theorem 2.7, that the fields of definition of the arithmetic lattices are conjugate:

Proof of Part (2) of Theorem 2.7. In the notation of the proof of the foregoing proposition, let $K'_1 = \iota_1(K_1)$. Consider the group $\Gamma'_1 := \iota_1(\overline{\Gamma}_1)$ as an arithmetic lattice of the group $\overline{\mathcal{G}}'_1 = {}^{\iota_1}\overline{\mathcal{G}}_1$ defined over the number field K'_1 . We have an algebraic isomorphism $\theta : \overline{\mathcal{G}}'_1 \times F \rightarrow \overline{\mathcal{G}}_2 \times F$ defined over F of the groups base changed to F . Further $\theta(\Gamma'_1)$ and $\overline{\Gamma}_2$ are elementwise conjugate in $\overline{\mathcal{G}}_2(F)$

By a theorem of Vinberg as given in Lemma 2.6 of [PR], it follows that the tracefields generated by $\text{Trace}(\text{Ad}(\gamma))$ for γ belonging to Γ'_1 (resp. Γ_2) generate

the field of definition K'_1 (resp. K_2) of the ambient group $\overline{\mathcal{G}}'_1$ (resp. $\overline{\mathcal{G}}_2$). Hence $K'_1 = K_2$ and this proves Part (2) of Theorem 2.7. \square

Henceforth, we will assume upto twisting the group scheme \mathcal{G}_1 by a field automorphism $\sigma : K_1 \rightarrow K'_1$, that $K := K_1 = K_2$ and both the group schemes \mathcal{G}_1 and \mathcal{G}_2 are defined over the same number field K .

3.4. Weak commensurability. We recall the concept of weak commensurability introduced by Gopal Prasad and Rapinchuk.

Definition 3.7. Let G_1 and G_2 be two semi-simple groups defined over a field F of characteristic zero. Two Zariski dense subgroups Γ_i of $G_i(F)$, for $i = 1, 2$ are said to be weakly commensurable if given any element of infinite order $\gamma_1 \in \Gamma_1$ (resp. $\gamma_2 \in \Gamma_2$) there exists an element of infinite order $\gamma_2 \in \Gamma_2$ (resp. $\gamma_1 \in \Gamma_1$) such that the subgroup of \bar{F}^* generated by the eigen values of γ_1 (resp. γ_2) (in a faithful representation of G_1) intersects nontrivially the subgroup generated by the eigenvalues of an element γ_2 (resp. γ_1).

It is clear that Proposition 3.6 implies that the lattices Γ_1 and Γ_2 are weakly commensurable. Although Theorem 2.7 follows now from the results proved by Gopal Prasad and A. Rapinchuk ([PR][Theorems 1 to 5]), the conclusion of Proposition 3.6 is stronger than the notion of weak commensurability. This leads us define a new relation on the class of arithmetic lattices, stronger than weak commensurability, which we call as characteristic equivalence. This notion allows us to simplify some of the arguments deducing commensurability type results from weak commensurability given in [PR].

4. CHARACTERISTIC EQUIVALENCE OF LATTICES

We define a notion of characteristic equivalence of lattices:

Definition 4.1. Let \mathcal{G}_1 (resp. \mathcal{G}_2) be algebraic groups defined respectively over a number field K . Let S_1 (resp. S_2) be a finite set of places respectively of K . Assume that for $i = 1, 2$, (K, \mathcal{G}_i, S_i) satisfy hypothesis **H1-H3**.

Let $\Gamma_1 \subset \mathcal{G}_1(K)$ (resp. $\Gamma_2 \subset \mathcal{G}_2(K)$) be S_1 (resp. S_2)-arithmetic subgroup of \mathcal{G}_1 (resp. \mathcal{G}_2).

We say that Γ_1 and Γ_2 are *characteristically equivalent* if given any semisimple element $\gamma_1 \in \Gamma_1$ (resp. $\gamma_2 \in \Gamma_2$) there exists a semisimple element $\gamma_2 \in \Gamma_2$ (resp. $\gamma_1 \in \Gamma_1$) such that the characteristic polynomials coincide,

$$P(\text{Ad}_{\mathcal{G}_1}(\gamma_1), x) = P(\text{Ad}_{\mathcal{G}_2}(\gamma_2), x).$$

Lemma 4.2. *In the definition of characteristic equivalence, we can further assume that the algebraic tori generated by γ_i in \mathcal{G}_i ($i = 1, 2$) are isogenous.*

Proof. Since the algebraic groups are absolutely almost simple, the isogeny class of the tori generated by γ_i is determined by the element $Ad_{\mathcal{G}_i}(\gamma_i)$ for $i = 1, 2$. Identifying the Lie algebras with K^N as vector spaces over K , the lemma follows. \square

We now establish Theorems 1 to 5 of [PR] under this stronger hypothesis of characteristic equivalence of lattices:

Theorem 4.3. *Let \mathcal{G}_1 (resp. \mathcal{G}_2) be algebraic groups defined respectively over a number field K . Let S_1 (resp. S_2) be a finite set of places respectively of K . Assume that for $i = 1, 2$, (K, \mathcal{G}_i, S_i) satisfy hypothesis **H1-H3**.*

Let $\Gamma_1 \subset \mathcal{G}_1(K)$ (resp. $\Gamma_2 \subset \mathcal{G}_2(K)$) be S_1 (resp. S_2)-arithmetic subgroup of \mathcal{G}_1 (resp. \mathcal{G}_2).

Suppose that Γ_1 and Γ_2 are characteristically equivalent lattices. Then the following holds:

- (1) *The groups \mathcal{G}_1 and \mathcal{G}_2 are of the same geometric type, or one of them is of type B_n and the other is of type C_n .*
- (2) *The set of isotropic places S_1^i and S_2^i coincide.*
- (3) *Assume further that \mathcal{G}_1 and \mathcal{G}_2 are of the same geometric type. If \mathcal{G}_1 is not of type $A_n, D_4, D_{2n+1}, (n > 1) E_6$, then the lattices Γ_1 and Γ_2 are commensurable.*
- (4) *In any characteristic equivalence class of arithmetic lattices, there are only finitely many commensurability classes of arithmetic lattices.*

Theorem 4.3 combined with Proposition 3.6 gives a proof of Theorem 2.7.

Remark 4.4. It is clear that characteristically equivalent lattices are weakly commensurable. In the exceptions when the groups are not commensurable, do there always exist pairs of characteristically equivalent lattices? It would be interesting to know whether characteristically equivalent lattices are (topologically) representation equivalent.

Let \mathcal{G} be a connected, absolutely almost simple algebraic group defined over K . Let T be a maximal K -torus in \mathcal{G} . Denote by Φ_T the root system of \mathcal{G} with respect to T , and by $W(\Phi_T)$ the Weyl group of Φ_T . Let L be the splitting field of T . There exists a natural injective homomorphism $\theta_T : \text{Gal}(L/K) \rightarrow \text{Aut}(\Phi_T)$.

For the proof of Theorem 4.3, we need the following theorem on the existence of irreducible tori ([PR2][Theorem 1]):

Theorem 4.5. *Let \mathcal{G} be a connected, absolutely almost simple algebraic group defined over a number field K . Suppose v is a place of K and T_v is a maximal K_v -torus of \mathcal{G} . Then there exists a K -torus T of \mathcal{G} such that it is conjugate to T_v by an element of $\mathcal{G}(K_v)$. Further, the image of θ_T contains the Weyl group $W(\Phi_T)$. In particular, T is an irreducible, anisotropic maximal K -torus of \mathcal{G} .*

The proof of this theorem is based on a theorem of A. Grothendieck that the variety of maximal tori is rational, and based on this a theorem of V. E.

Voskresenskii showing that the Galois group of the splitting field of the generic maximal tori contains the Weyl group.

Corollary 4.6. *With notation as in Theorem 4.5, let Γ be a S -arithmetic lattice in \mathcal{G} and $v \in S^i$. Assume further that T_v is an isotropic torus. Then there exists an element $\gamma \in \Gamma$ which generates T over K .*

Proof. By [PIR, Theorem 5.12], there exists non-torsion elements in $T_1(\mathcal{O}_K(S))$. Since $\Gamma \cap T(\mathcal{O}_K(S))$ is of finite index in $T(\mathcal{O}_K(S))$, there exists a non-torsion element $\gamma \in \Gamma \cap T(\mathcal{O}_K(S))$. Since T is irreducible, γ will generate T over K . \square

Proof of Part (1) of Theorem 4.3. The equality of the characteristic polynomials with respect to the adjoint representation implies that the dimensions of the Lie algebras are equal. If the algebraic groups involved are not of type B_6 , C_6 or E_6 , then the geometric type is determined by the dimension of the Lie algebra.

For the proof of Part (1) in this exceptional case, we argue as in proof of Theorem 1 in [PR, page 130]: by Corollary 4.6, choose a torus T_1 and an element $\gamma_1 \in \Gamma_1$ which generates T_1 over K . By characteristic equivalence, there exists an element $\gamma_2 \in \Gamma_2$ having the same characteristic polynomial as γ_1 . By Lemma 4.2, we can further assume that the tori T_1 and T_2 generated respectively by γ_1 and γ_2 are isogenous over K .

Let L be the splitting field of T_1 (equivalently of T_2). By Theorem 4.5, the image of θ_{T_1} contains the Weyl group $W(\Phi_{T_1})$. We can assume that the geometric type of \mathcal{G}_1 is of type either B_6 or C_6 . In this case, all automorphisms of Φ_{T_1} are inner, and the cardinality of $\text{Gal}(L/K)$ is thus equal to $|W(\Phi_{T_1})|$. From the injectivity of the map θ_{T_2} , we see that $|W(\Phi_{T_1})|$ divides the cardinality of $\text{Aut}(\Phi_{T_2})$. But the cardinality of $W(B_6)$ is $2^{10}3^{25}$, whereas the cardinality of $\text{Aut}(E_6)$ is given by 2^73^45 . This implies that \mathcal{G}_2 cannot be of type E_6 . \square

Proof of Part (2) of Theorem 4.3. This is the analogue of Theorem 3 of [PR], and we follow the proof as given in [PR, page 139] of this theorem. If $v \in S_1^i$ is a place where \mathcal{G}_1 is isotropic, choose a maximal split K_v -torus $T_{1,v}$ of \mathcal{G}_1 . By Theorem 4.5, there exists a K -irreducible anisotropic maximal K -torus T_1 of \mathcal{G}_1 such that it is conjugate to $T_{1,v}$ by an element of $\mathcal{G}_1(K_v)$. Since T_1 is anisotropic the quotient $T_{1,S_1}/T_1(\mathcal{O}_K(S_1))$ is compact where $T_{1,S_1} = \prod_{v \in S_1} T_1(K_v)$. This implies that the quotient $T_1(K_v)/\mathcal{C}$ is also compact, where \mathcal{C} is the closure of $T_1(\mathcal{O}_K(S_1))$ in $T_1(K_v)$. Since T_1 is K_v -isotropic, \mathcal{C} is noncompact. The closure of $\text{Ad}(T_1(\mathcal{O}_K(S_1)))$ inside $GL_N(K_v)$ will also be noncompact. Since $T_1(\mathcal{O}_K(S_1))$ is a finitely generated abelian group consisting of semisimple elements, it can be simultaneously diagonalised over \overline{K}_v . If the eigenvalues of every element in $T_1(\mathcal{O}_K(S_1))$ is a v -adic unit, then this implies that the closure of $T_1(\mathcal{O}_K(S_1))$ is compact, contradicting our earlier conclusion. Since $\Gamma_1 \cap T_1(\mathcal{O}_K(S_1))$ is of finite index in $T_1(\mathcal{O}_K(S_1))$, there exists an element $\gamma_1 \in \Gamma_1$ such that at least one

eigenvalue of $Ad_{\mathcal{G}_1}(\gamma_1) \in GL_N$ is not a v -adic unit. By assumption there exists an element $\gamma_2 \in \Gamma_2$ which is characteristic equivalent to γ_1 .

If $v \notin S_2^i$, then the closure of the subgroup $\mathcal{G}_2(\mathcal{O}_K(S_2))$ in $\mathcal{G}_2(K_v)$ is compact. But this implies that all the eigenvalues of $Ad_{\mathcal{G}_2}(\gamma_2)$ are v -adic units. This yields a contradiction and hence $S_1^i \subset S_2^i$. By symmetry we get $S_1^i = S_2^i$. \square

Remark 4.7. It is known that weak approximation holds for the tori constructed in Theorem 4.5 (see [PR2]). One could have also used this fact to give a slight variation of the above argument.

We now prove Theorem 6.2 of [PR]. It is the basic input needed to prove Parts (3) and (4) of Theorem 4.3.

Theorem 4.8. *With hypothesis as in Theorem 4.3, for any place v of K ,*

$$\mathrm{rk}_{K_v} \mathcal{G}_1 = \mathrm{rk}_{K_v} \mathcal{G}_2$$

Proof. Let $T_{1,v}$ be a maximal K_v -split torus of \mathcal{G}_1 , and choose a K -torus T_1 and an element $\gamma_1 \in \Gamma_1$ as in Corollary 4.6. By the characteristic equivalence of Γ_1 and Γ_2 , there exists an element $\gamma_2 \in \Gamma_2$ for which there is an equality of characteristic polynomials

$$P(Ad_{\mathcal{G}_1}(\gamma_1), x) = P(Ad_{\mathcal{G}_2}(\gamma_2), x).$$

This implies that the elements $Ad_{\mathcal{G}_1}(\gamma_1)$ and $Ad_{\mathcal{G}_2}(\gamma_2)$ considered as elements in GL_N/K are conjugate, and hence generate isomorphic tori over K . Let T_2 be the tori generated by γ_2 . We have,

$$\mathrm{rk}_{K_v} \mathcal{G}_1 = \mathrm{rk}_{K_v} T_1 = \mathrm{rk}_{K_v} T_2 \leq \mathrm{rk}_{K_v} \mathcal{G}_2$$

By symmetry, this proves the theorem. \square

The proofs of Part (3) and (4) of Theorem 4.3 follow as in page 147-148 of [PR]. For the sake of completeness, we give a brief outline of the proof.

Proof of Part (3) of Theorem 4.3. If the geometric type is of type D_{2n} , ($n > 2$) this is proved in [PR3].

If the geometric type is not of A , D or E_6 type, the equality of local ranks implies that $\overline{\mathcal{G}}_{1,v} \simeq \overline{\mathcal{G}}_{2,v}$ for any place v of K . For archimedean places, this follows from classification results [T]. For a non-archimedean place, this follows from the fact that there can be at most two possible forms for the adjoint group. To see the latter fact, we observe that the centre Z of the simply connected cover of \mathcal{G} is a subgroup of μ_2 , where \mathcal{G} is not of type A , D , E_6 . From the equality of the Galois cohomology groups, we get that $H^1(K_v, \overline{\mathcal{G}}) \simeq H^2(K_v, Z)$, which can be identified with a subgroup of the 2-torsion in the Brauer group of K_v . Since this is of cardinality two, and the outer automorphism group is trivial, this implies that there are at most two forms of $\overline{\mathcal{G}}$ for any non-archimedean place v . Hence an equality of ranks over K_v implies that the forms are isomorphic.

Now Part (3) of Theorem 4.3 follows from the Hasse principle, viz., the injectivity of the localization map,

$$H^1(K, \overline{\mathcal{G}}) \rightarrow \bigoplus_v H^1(K_v, \overline{\mathcal{G}})$$

where v runs over all places of K . □

Proof of Part (4) of Theorem 4.3. From Theorem 4.8, it can be seen by a Chebotarev density argument ([PR, Theorem 6.3] that the minimal splitting field L_i over which \mathcal{G}_i becomes the inner form of a split group for $i = 1, 2$ coincide. Moreover, the set of places V_i at which \mathcal{G}_i is not quasi-split coincide.

Fixing the geometric type, say a split form \mathcal{G}_0 over K of adjoint type, the groups are parametrized by cocycles $c \in H^1(K, \text{Aut}(\mathcal{G}_0))$. Consider the exact sequence,

$$1 \rightarrow \mathcal{G}_0 \rightarrow \text{Aut}(\mathcal{G}_0) \rightarrow \text{Out}(\mathcal{G}_0) \rightarrow 1.$$

This yields an exact sequence,

$$H^1(K, \mathcal{G}_0) \rightarrow H^1(K, \text{Aut}(\mathcal{G}_0)) \rightarrow H^1(K, \text{Out}(\mathcal{G}_0)).$$

The condition that the group becomes an inner form over L implies that this cocycle lies in the image of $H^1(G(L/K), \text{Out}(\mathcal{G}_0))$ which is a finite group. Now the required finiteness follows from the finiteness of the Hasse principle, i.e., the kernel of the localization map,

$$H^1(K, \overline{\mathcal{G}}_0) \rightarrow \bigoplus_{v \notin V} H^1(K_v, \overline{\mathcal{G}}_0),$$

where $V = V_i$ is a fixed finite set of places of K . □

Remark 4.9. It is further deduced [PR, Theorem 6], that if \mathcal{G}_1 and \mathcal{G}_2 have the same geometric type satisfying the hypothesis of Theorem 4.8, then the Tits indices are equal at all places of K .

Remark 4.10. It is possible to consider modifications of the concept of characteristic equivalence, say more generally on the class of subgroups not necessarily arithmetic lattices: for example one can consider the equality of the characteristic polynomials on ‘big’ subsets, like subgroups of finite index, or Zariski open, or even some kind of Hilbertian sets.

Yet another relation that can be imposed is to define two lattices to be trace equivalent if the set of traces of elements with respect to the adjoint representation coincide for the two lattices.

It would be interesting to know whether these properties would imply commensurability results. To conclude commensurability type results will require analogues of Theorems 4.5 or 4.8.

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